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Chapter 7

The Normal Distribution

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Outline

- **7-1 Introduction**
- **7-2 Properties of the Normal Distribution**
- **7-3 The Standard Normal Distribution**
- **7-4 Applications of the Normal Distribution**

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Outline

- **7-5 The Central Limit Theorem**
- **7-6 The Normal Approximation to the Binomial Distribution**

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Objectives

- **Identify distributions as symmetric or skewed.**
- **Identify the properties of the normal distribution.**
- **Find the area under the standard normal distribution given various z values.**

Objectives

- Find probabilities for a normally distributed variable by transforming it into a standard normal variable.
- Find specific data values for given percentages using the standard normal distribution.

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Objectives

- **Use the Central Limit Theorem to solve problems involving sample means.**
- **Use the normal approximation to compute probabilities for a binomial variable.**

7-2 Properties of the Normal Distribution

- Many continuous variables have distributions that are bell-shaped and are called **approximately normally distributed variables**.
- The theoretical curve, called the **normal distribution curve**, can be used to study many variables that are not normally distributed but are approximately normal.

7-2 Mathematical Equation for the Normal Distribution

The mathematical equation for the normal distribution:

$$y = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$$

where

$e \approx 2.718$

$\pi \approx 3.14$

$\mu = \text{population mean}$

$\sigma = \text{population standard deviation}$

7-2 Properties of the Normal Distribution

- The shape and position of the normal distribution curve depend on two parameters, the mean and the standard deviation.
- Each normally distributed variable has its own normal distribution curve, which depends on the values of the variable's mean and standard deviation.

7-2 Properties of the Theoretical Normal Distribution

- **The normal distribution curve is bell-shaped.**
- **The mean, median, and mode are equal and located at the center of the distribution.**
- **The normal distribution curve is unimodal (single mode).**

7-2 Properties of the Theoretical Normal Distribution

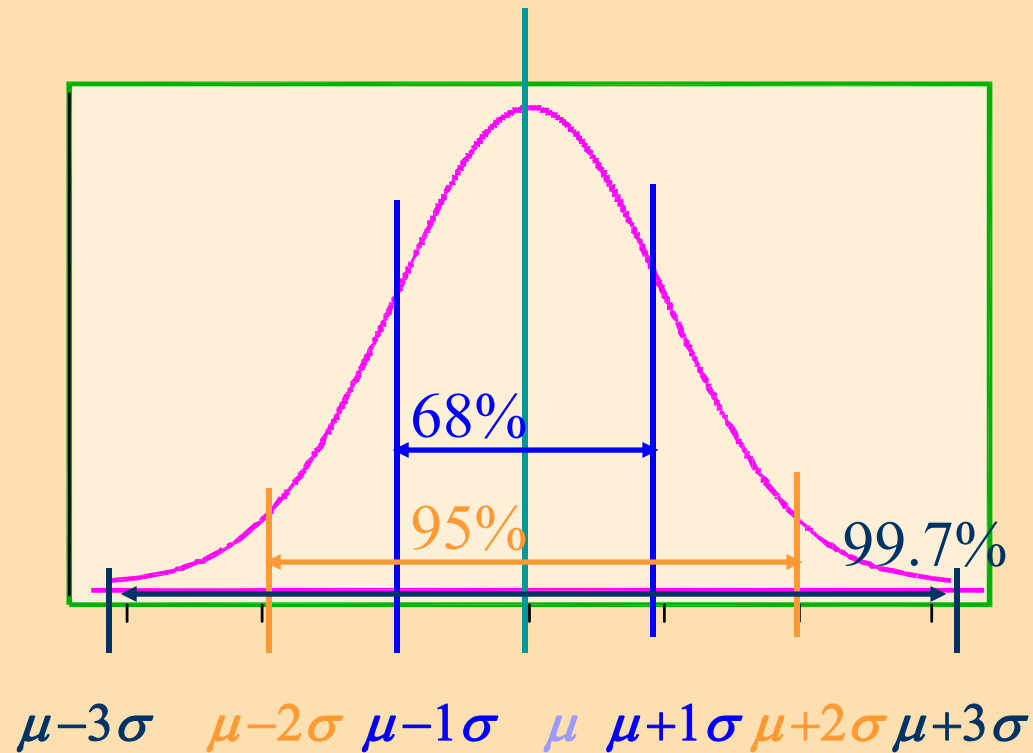
- **The curve is symmetrical about the mean.**
- **The curve is continuous.**
- **The curve never touches the x -axis.**
- **The total area under the normal distribution curve is equal to 1.**

7-2 Properties of the Theoretical Normal Distribution

- **The area under the normal curve that lies within**
 - ✓ **one standard deviation of the mean is approximately 0.68 (68%).**
 - ✓ **two standard deviations of the mean is approximately 0.95 (95%).**
 - ✓ **three standard deviations of the mean is approximately 0.997 (99.7%).**

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7-2 Areas Under the Normal Curve



7-3 The Standard Normal Distribution

- The **standard normal distribution** is a normal distribution with a mean of 0 and a standard deviation of 1.
- All normally distributed variables can be transformed into the standard normally distributed variable by using the formula for the standard score:
(see next slide)

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7-3 The Standard Normal Distribution

$$z = \frac{\text{value} - \text{mean}}{\text{standard deviation}}$$

or

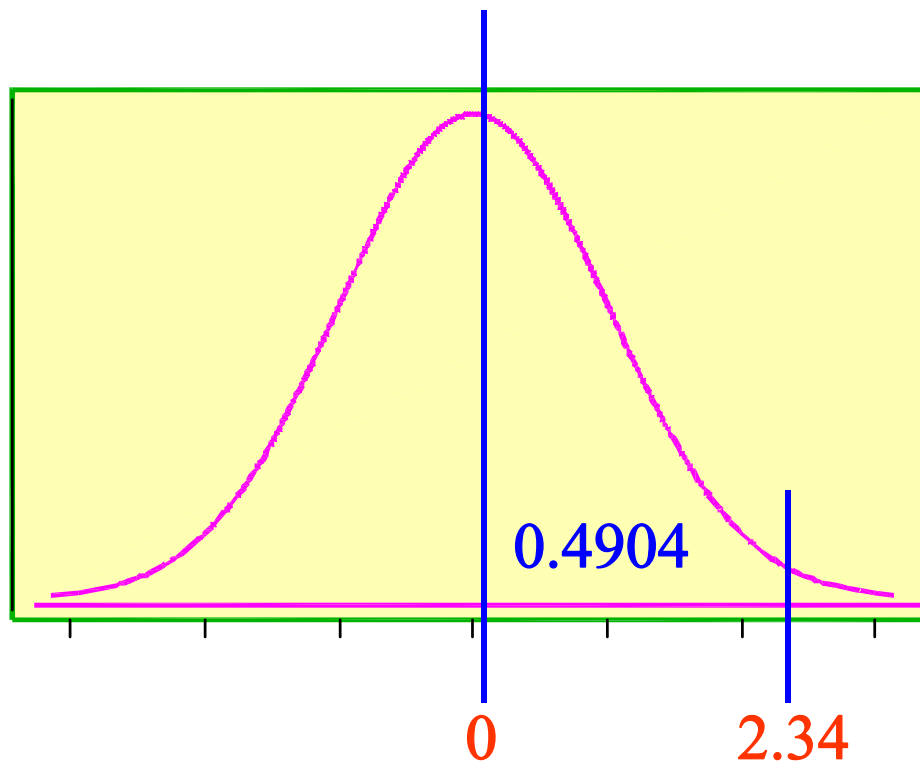
$$z = \frac{X - \mu}{\sigma}$$

7-3 Area Under the Standard Normal Curve - Example

- Find the area under the standard normal curve between $z = 0$ and $z = 2.34 \Rightarrow P(0 \leq z \leq 2.34)$.
- Use your table at the end of the text to find the area.
- The next slide shows the shaded area.

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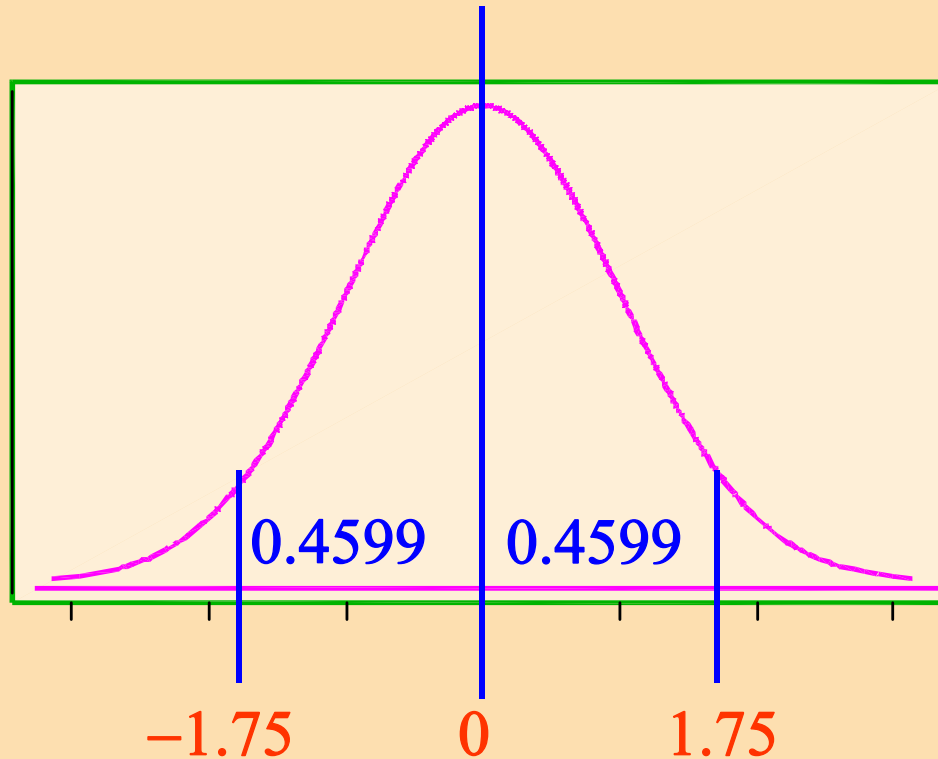
7-3 Area Under the Standard Normal Curve - Example



7- 3 Area Under the Standard Normal Curve - Example

- Find the area under the standard normal curve between $z = 0$ and $z = -1.75 \Rightarrow P(-1.75 \leq z \leq 0)$.
- Use the symmetric property of the normal distribution and your table at the end of the text to find the area.
- The next slide shows the shaded area.

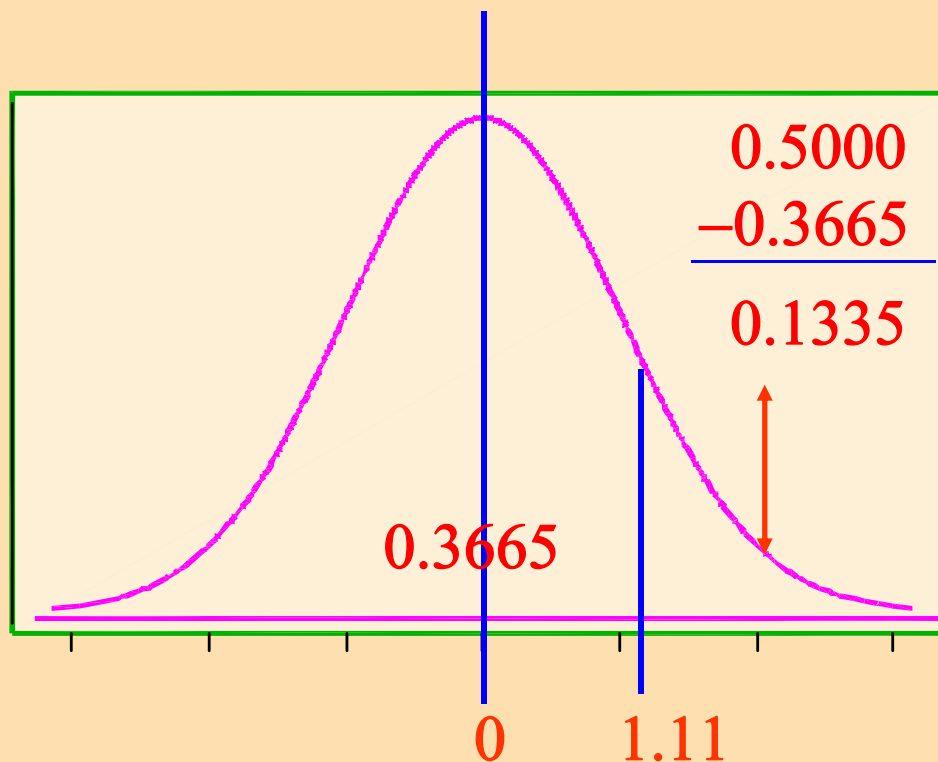
7-3 Area Under the Standard Normal Curve - Example



7-3 Area Under the Standard Normal Curve - Example

- Find the area to the right of $z = 1.11$
 $\Rightarrow P(z > 1.11)$.
- Use your table at the end of the text to find the area.
- The next slide shows the shaded area.

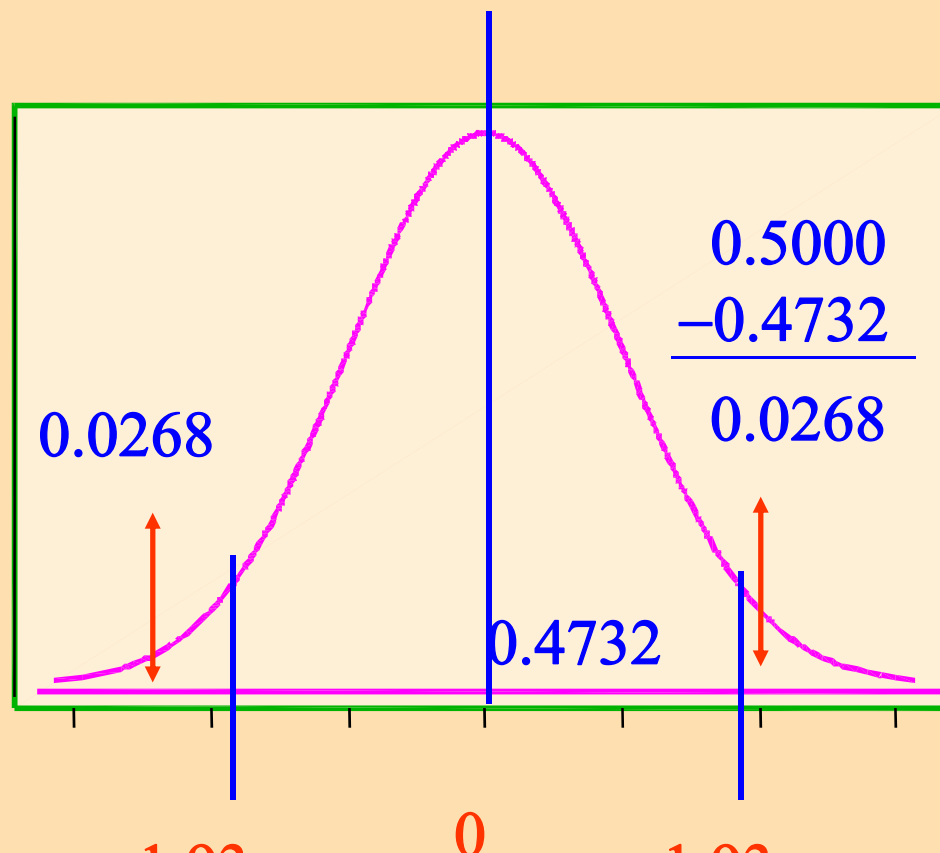
7-3 Area Under the Standard Normal Curve - Example



7-3 Area Under the Standard Normal Curve - Example

- Find the area to the left of $z = -1.93$
 $\Rightarrow P(z < -1.93)$.
- Use the symmetric property of the normal distribution and your table at the end of the text to find the area.
- The next slide shows the area.

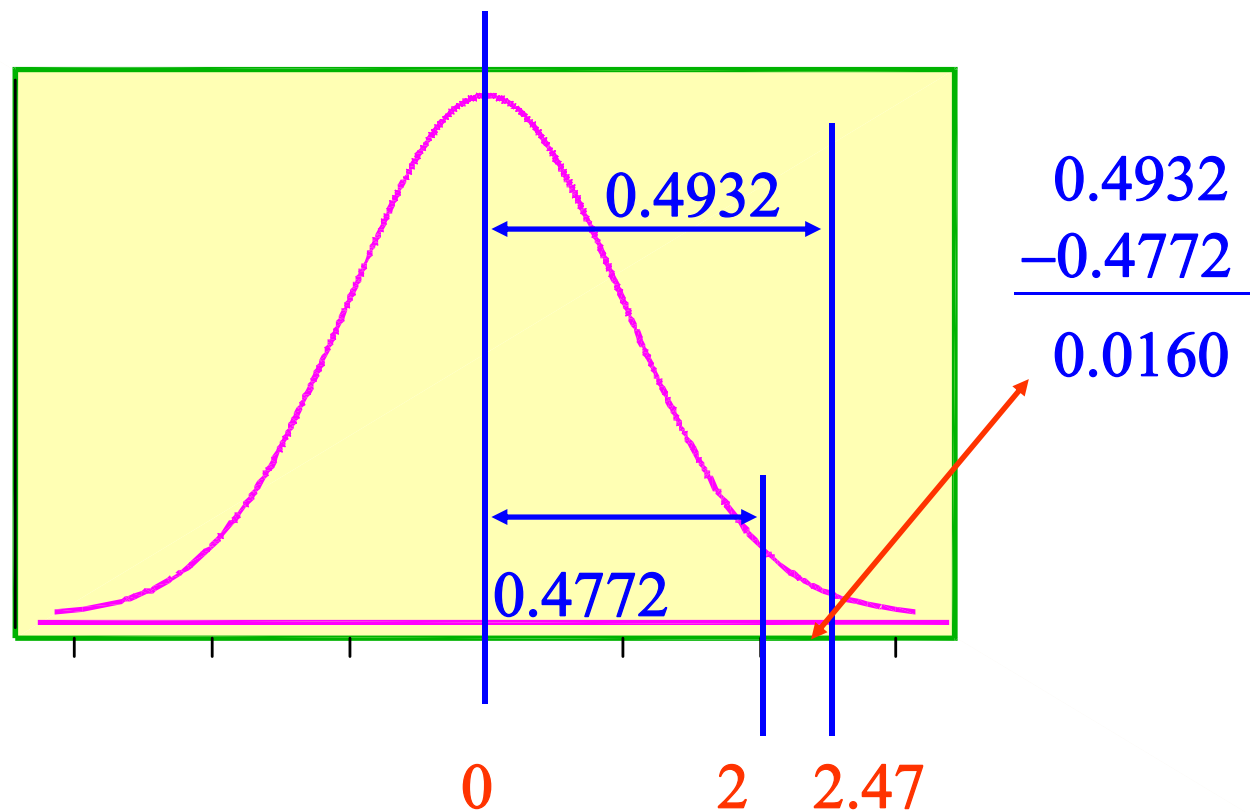
7-3 Area Under the Standard Normal Curve - Example



7-3 Area Under the Standard Normal Curve - Example

- Find the area between $z = 2$ and $z = 2.47 \Rightarrow P(2 \leq z \leq 2.47)$.
- Use the symmetric property of the normal distribution and your table at the end of the text to find the area.
- The next slide shows the area.

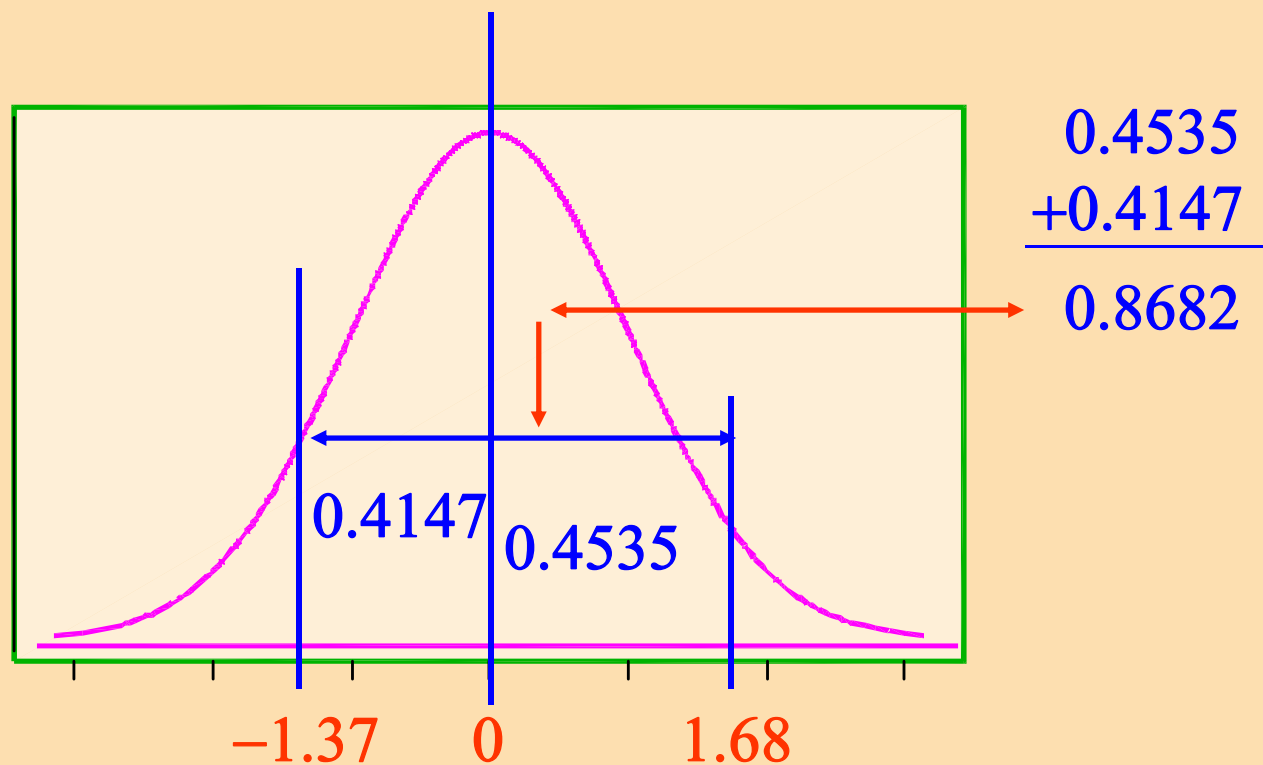
7-3 Area Under the Standard Normal Curve - Example



7-3 Area Under the Standard Normal Curve - Example

- Find the area between $z = 1.68$ and $z = -1.37 \Rightarrow P(-1.37 \leq z \leq 1.68)$.
- Use the symmetric property of the normal distribution and your table at the end of the text to find the area.
- The next slide shows the area.

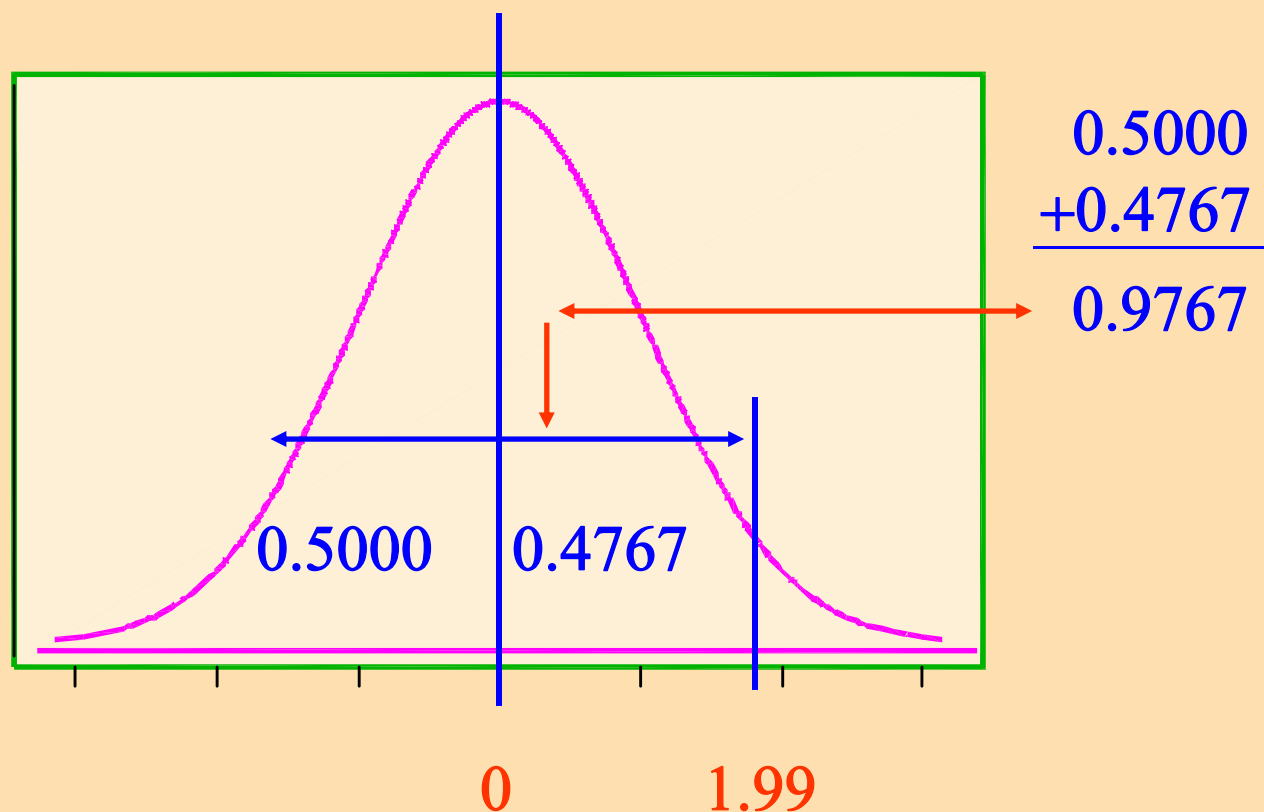
7-3 Area Under the Standard Normal Curve - Example



7-3 Area Under the Standard Normal Curve - Example

- Find the area to the left of $z = 1.99$
 $\Rightarrow P(z < 1.99)$.
- Use your table at the end of the text to find the area.
- The next slide shows the area.

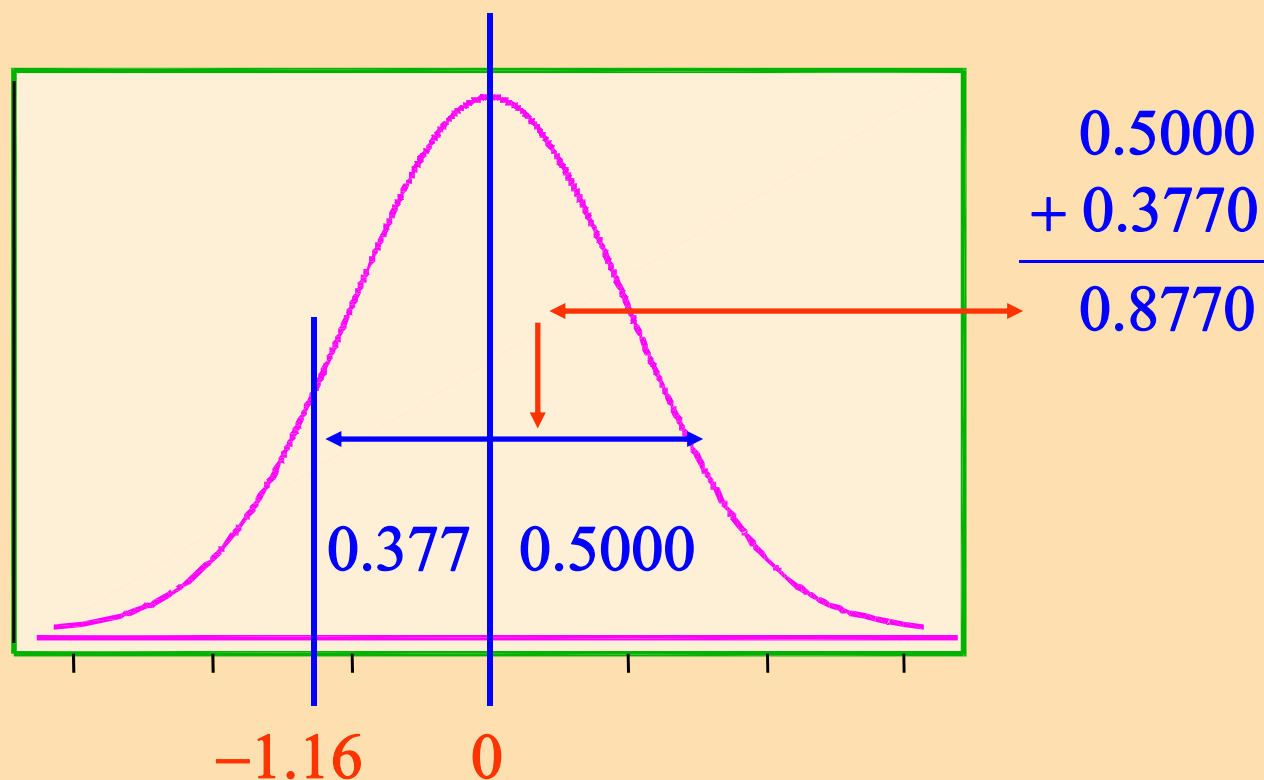
7-3 Area Under the Standard Normal Curve - Example



7-3 Area Under the Standard Normal Curve - Example

- Find the area to the right of $z = -1.16 \Rightarrow P(z > -1.16)$.
- Use your table at the end of the text to find the area.
- The next slide shows the area.

7-3 Area Under the Standard Normal Curve - Example



RECALL: The Standard Normal Distribution

$$z = \frac{\text{value} - \text{mean}}{\text{standard deviation}}$$

or

$$z = \frac{X - \mu}{\sigma}$$

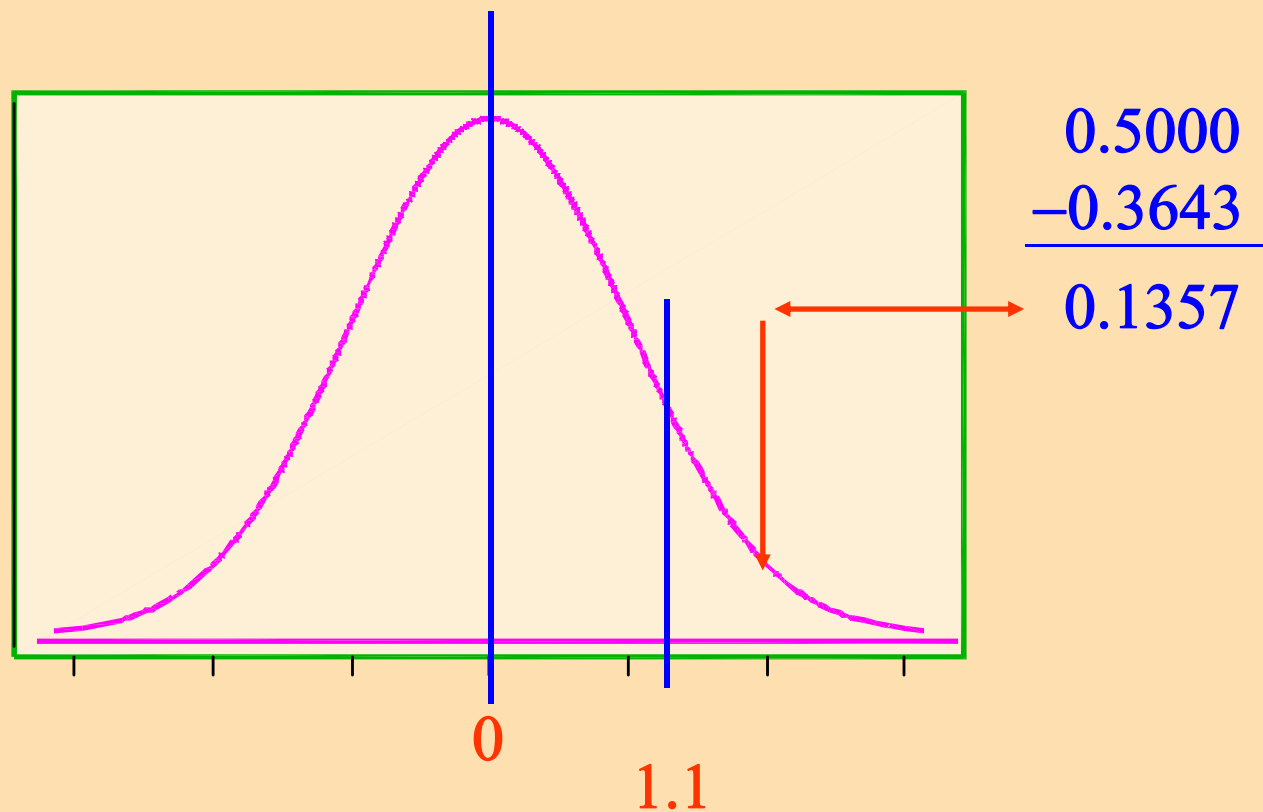
7-4 Applications of the Normal Distribution - Example

- Each month, an American household generates an average of 28 pounds of newspaper for garbage or recycling. Assume the standard deviation is 2 pounds. Assume the amount generated is normally distributed.
- If a household is selected at random, find the probability of its generating:

7-4 Applications of the Normal Distribution - Example

- More than 30.2 pounds per month.
- First find the z-value for 30.2.
 $z = [X - \mu] / \sigma = [30.2 - 28] / 2 = 1.1.$
- Thus, $P(z > 1.1) = 0.5 - 0.3643 = 0.1357.$
- That is, the probability that a randomly selected household will generate more than 30.2 lbs. of newspapers is 0.1357 or 13.57%.

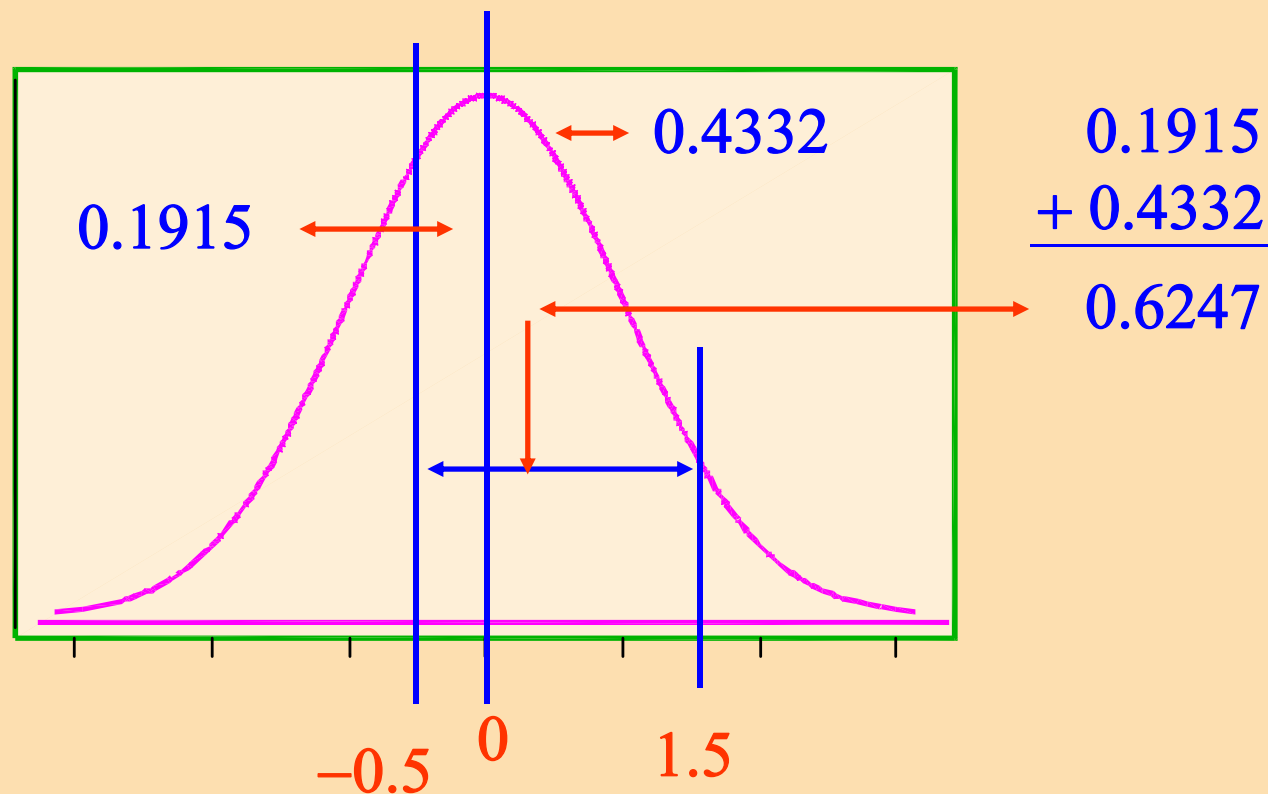
7-4 Applications of the Normal Distribution - Example



7-4 Applications of the Normal Distribution - Example

- Between 27 and 31 pounds per month.
- First find the z-value for 27 and 31.
 $z_1 = [X - \mu] / \sigma = [27 - 28] / 2 = -0.5;$
 $z_2 = [31 - 28] / 2 = 1.5$
- Thus, $P(-0.5 \leq z \leq 1.5) = 0.1915 + 0.4332 = 0.6247.$

7-4 Applications of the Normal Distribution - Example



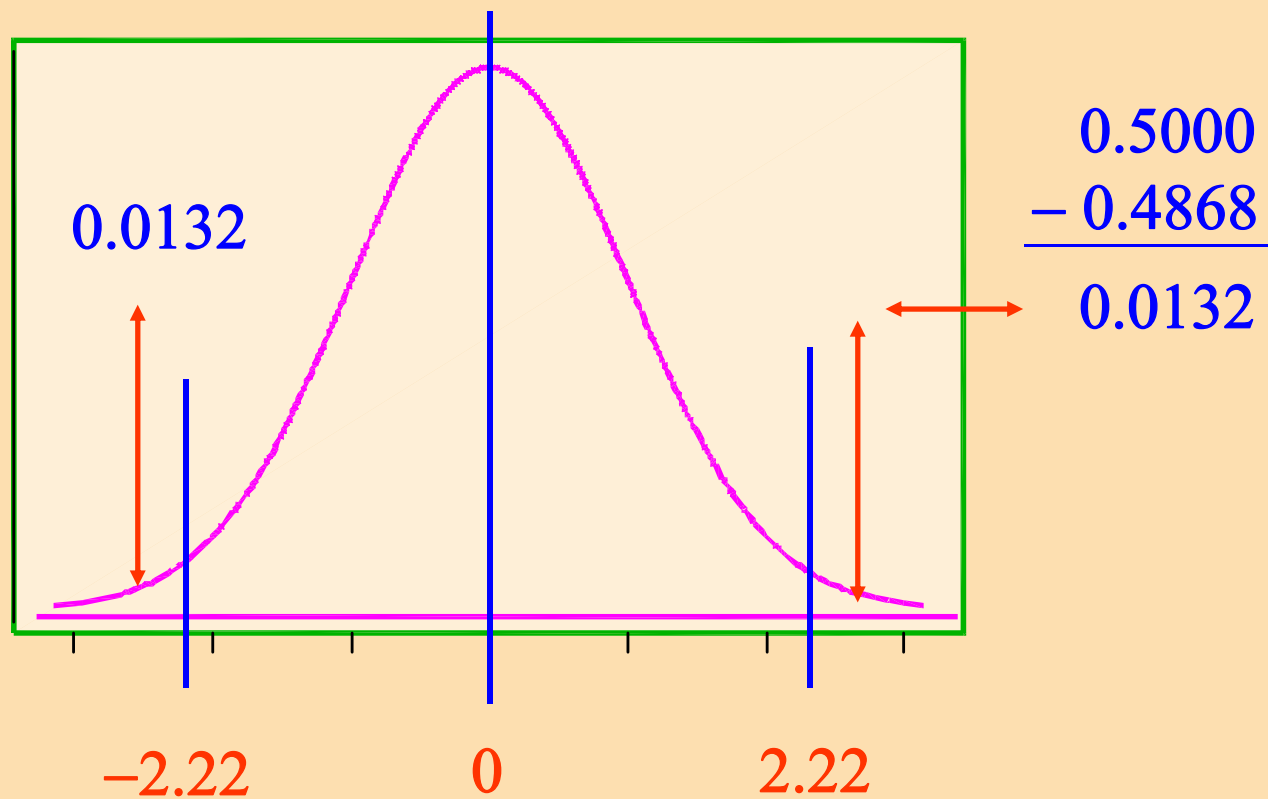
7-4 Applications of the Normal Distribution - Example

- The American Automobile Association reports that the average time it takes to respond to an emergency call is 25 minutes. Assume the variable is approximately normally distributed and the standard deviation is 4.5 minutes. If 80 calls are randomly selected, approximately how many will be responded to in less than 15 minutes?

7-4 Applications of the Normal Distribution - Example

- First find the z-value for 15 is
 $z = [X - \mu] / \sigma = [15 - 25] / 4.5 = -2.22.$
- Thus, $P(z < -2.22) = 0.5000 - 0.4868 = 0.0132.$
- The number of calls that will be made in less than 15 minutes = $(80)(0.0132) = 1.056 \approx 1.$

7-4 Applications of the Normal Distribution - Example



7-4 Applications of the Normal Distribution - Example

- An exclusive college desires to accept only the top 10% of all graduating seniors based on the results of a national placement test. This test has a mean of 500 and a standard deviation of 100. Find the cutoff score for the exam. Assume the variable is normally distributed.

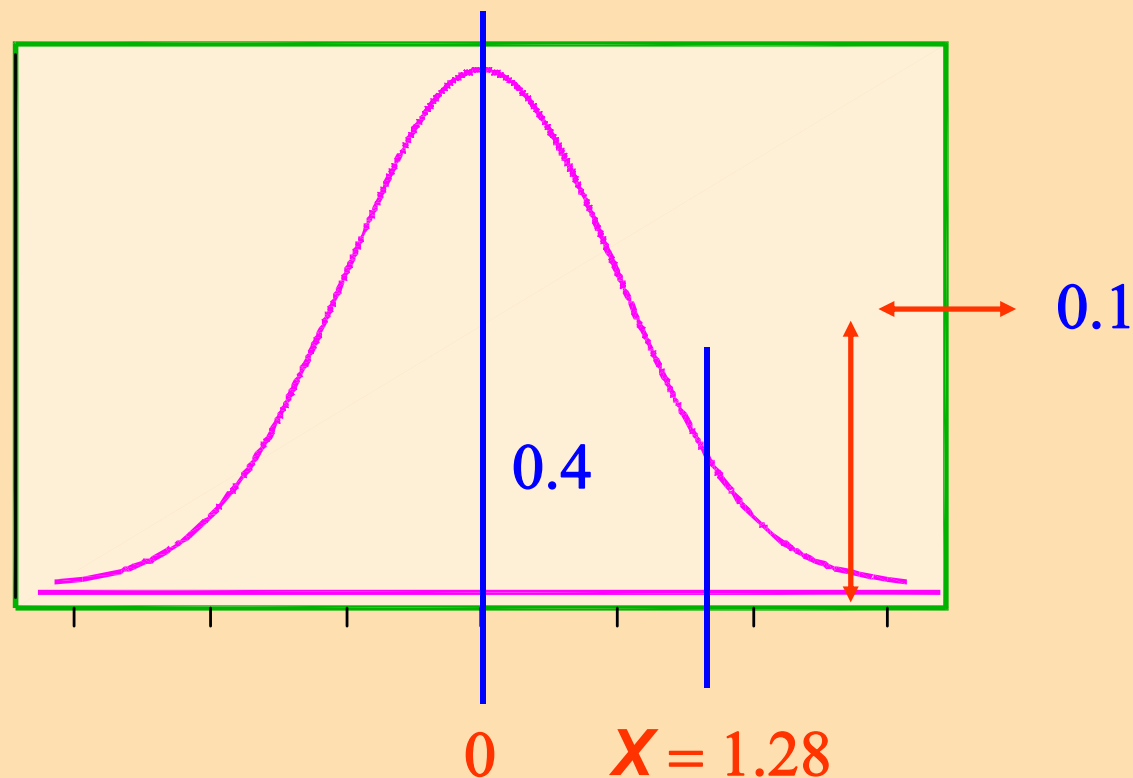
7- 4 Applications of the Normal Distribution - **Example**

- **Work backward to solve this problem.**
- **Subtract 0.1 (10%) from 0.5 to get the area under the normal curve for accepted students.**
- **Find the z value that corresponds to an area of 0.4000 by looking up 0.4000 in the area portion of Table E. Use the closest value, 0.3997.**

7- 4 Applications of the Normal Distribution - Example

- Substitute in the formula $z = \frac{X - \mu}{\sigma}$ and solve for X .
- The z -value for the cutoff score (X) is $z = [X - \mu]/\sigma = [X - 500]/100 = 1.28$. (See next slide).
- Thus, $X = (1.28)(100) + 500 = 628$.
- The score of 628 should be used as a cutoff score.

7-4 Applications of the Normal Distribution - Example



7-4 Applications of the Normal Distribution - Example

- **NOTE:** To solve for X , use the following formula: $X = z \cdot \sigma + \mu$.
- **Example:** For a medical study, a researcher wishes to select people in the middle 60% of the population based on blood pressure. (Continued on the next slide).

7-4 Applications of the Normal Distribution - Example

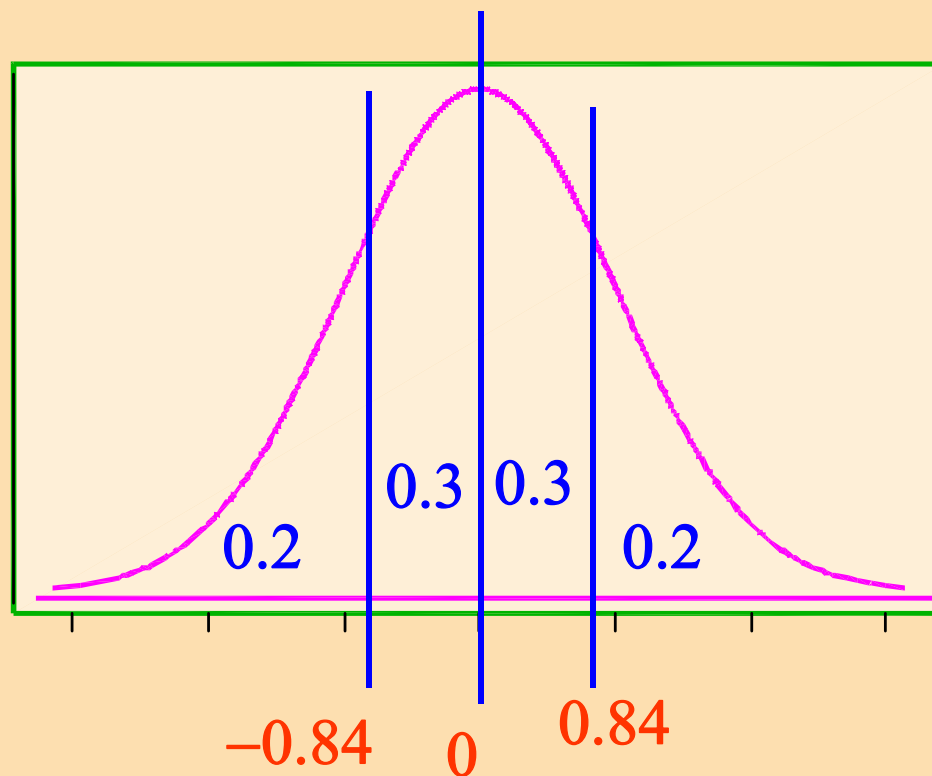
- (Continued)-- If the mean systolic blood pressure is 120 and the standard deviation is 8, find the upper and lower readings that would qualify people to participate in the study.

7-4 Applications of the Normal Distribution - Example

(continued)

- Note that two values are needed, one above the mean and one below the mean. The closest z values are 0.84 and -0.84 respectively.
- $X = (z)(\sigma) + \mu = (0.84)(8) + 120 = 126.72$.
The other $X = (-0.84)(8) + 120 = 113.28$.
See next slide.
- i.e. the middle 60% of BP readings is between 113.28 and 126.72.

7-4 Applications of the Normal Distribution - Example



9 7-5 Distribution of Sample Means

- **Distribution of Sample means:** A sampling distribution of sample means is a distribution obtained by using the means computed from random samples of a specific size taken from a population.

7-5 Distribution of Sample Means

- **Sampling error** is the difference between the sample measure and the corresponding population measure due to the fact that the sample is not a perfect representation of the population.

7-5 Properties of the Distribution of Sample Means

- **The mean of the sample means will be the same as the population mean.**
- **The standard deviation of the sample means will be smaller than the standard deviation of the population, and it will be equal to the population standard deviation divided by the square root of the sample size.**

7-5 Properties of the Distribution of Sample Means - Example

- Suppose a professor gave an 8-point quiz to a small class of four students. The results of the quiz were 2, 6, 4, and 8. Assume the four students constitute the population.
- The mean of the population is
 $\mu = (2 + 6 + 4 + 8)/4 = 5.$

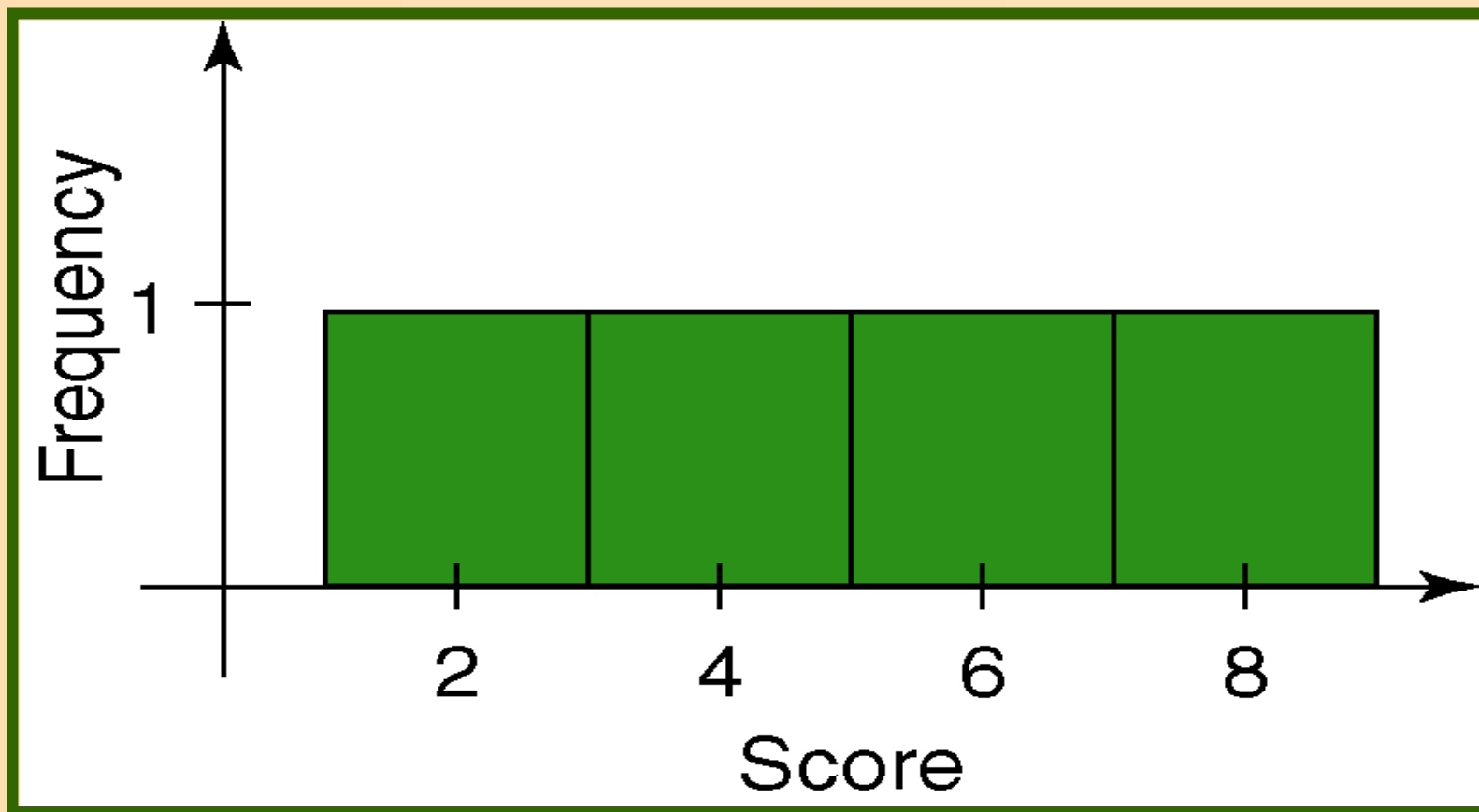
7-5 Properties of the Distribution of Sample Means - **Example**

- The standard deviation of the population is

$$\sigma = \sqrt{\left\{ \left[(2-5)^2 + (6-5)^2 + (4-5)^2 + (8-5)^2 \right] / 4 \right\}}$$
$$= 2.236.$$

- The graph of the distribution of the scores is uniform and is shown on the next slide.
- Next we will consider all samples of size 2 taken with replacement.

7-5 Graph of the Original Distribution



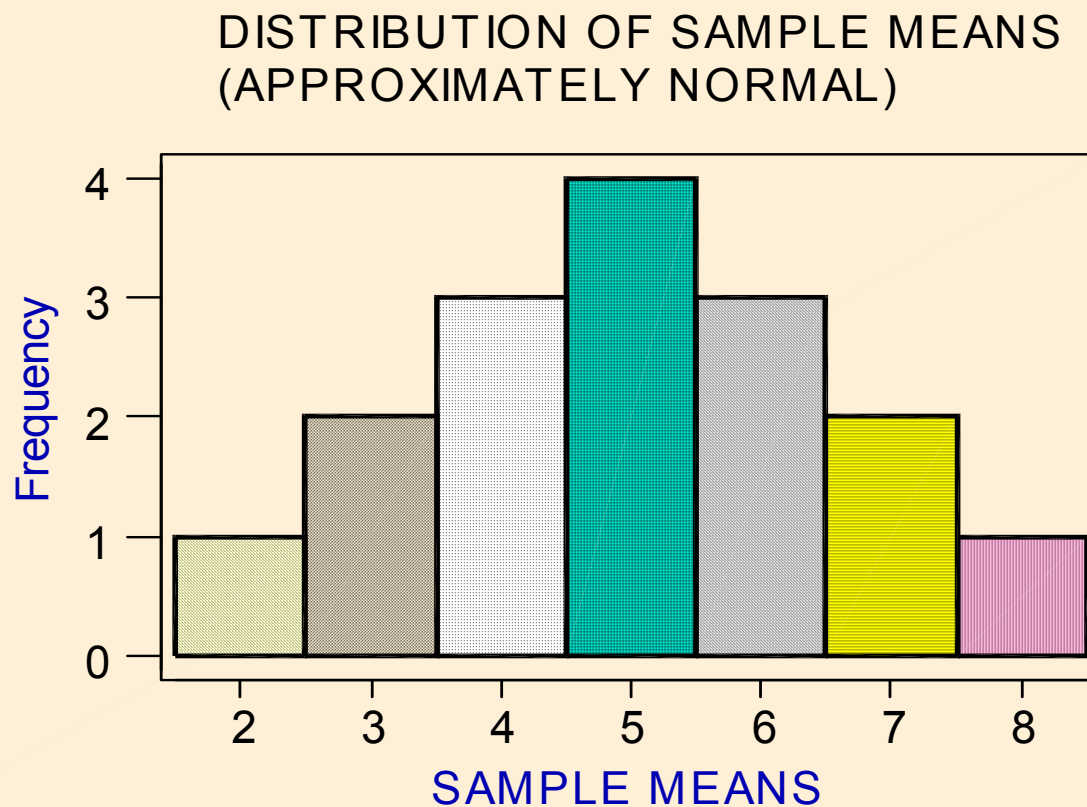
7-5 Properties of the Distribution of Sample Means - Example

Sample	Mean	Sample	Mean
2, 2	2	6, 2	4
2, 4	3	6, 4	5
2, 6	4	6, 6	6
2, 8	5	6, 8	7
4, 2	3	8, 2	5
4, 4	4	8, 4	6
4, 6	5	8, 6	7
4, 8	6	8, 8	8

7-5 Frequency Distribution of the Sample Means - Example

<i>\bar{X}</i> (mean)	2	3	4	5	6	7	8
<i>f</i>	1	2	3	4	3	2	1

7-5 Graph of the Sample Means



7-5 Mean and Standard Deviation of the Sample Means

Mean of Sample Means

$$\mu_{\bar{x}} = \frac{2 + 3 + \dots + 8}{16} = \frac{80}{16} = 5$$

which is the same as the

population mean. Thus $\mu_{\bar{x}} = \mu$.

7-5 Mean and Standard Deviation of the Sample Means

The standard deviation of the sample means is

$$\sigma_{\bar{x}} = \sqrt{\frac{(2-5)^2 + (3-5)^2 + \dots + (8-5)^2}{16}}$$
$$= 1.581.$$

This is the same as $\frac{\sigma}{\sqrt{2}}$.

7-5 The Standard Error of the Mean

The standard deviation of the sample means is called the standard error of the mean. Hence

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}.$$

7-5 The Central Limit Theorem

- As the sample size n increases, the shape of the distribution of the sample means taken from a population with mean μ and standard deviation of σ will approach a normal distribution. As previously shown, this distribution will have a mean μ and standard deviation σ/\sqrt{n} .

7-5 The Central Limit Theorem

The central limit theorem can be used to answer questions about sample means in the same manner that the normal distribution can be used to answer questions about individual values. The only difference is that a new formula must be used for the z - values. It is

$$z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}.$$

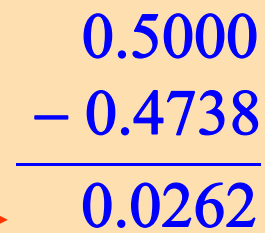
7-5 The Central Limit Theorem - Example

- **A.C. Nielsen reported that children between the ages of 2 and 5 watch an average of 25 hours of TV per week. Assume the variable is normally distributed and the standard deviation is 3 hours. If 20 children between the ages of 2 and 5 are randomly selected, find the probability that the mean of the number of hours they watch TV is greater than 26.3 hours.**

7-5 The Central Limit Theorem - Example

- The standard deviation of the sample means is $\sigma/\sqrt{n} = 3/\sqrt{20} = 0.671$.
- The z-value is $z = (26.3 - 25)/0.671 = 1.94$.
- Thus $P(z > 1.94) = 0.5 - 0.4738 = 0.0262$.
That is, the probability of obtaining a sample mean greater than 26.3 is $0.0262 = 2.62\%$.

5



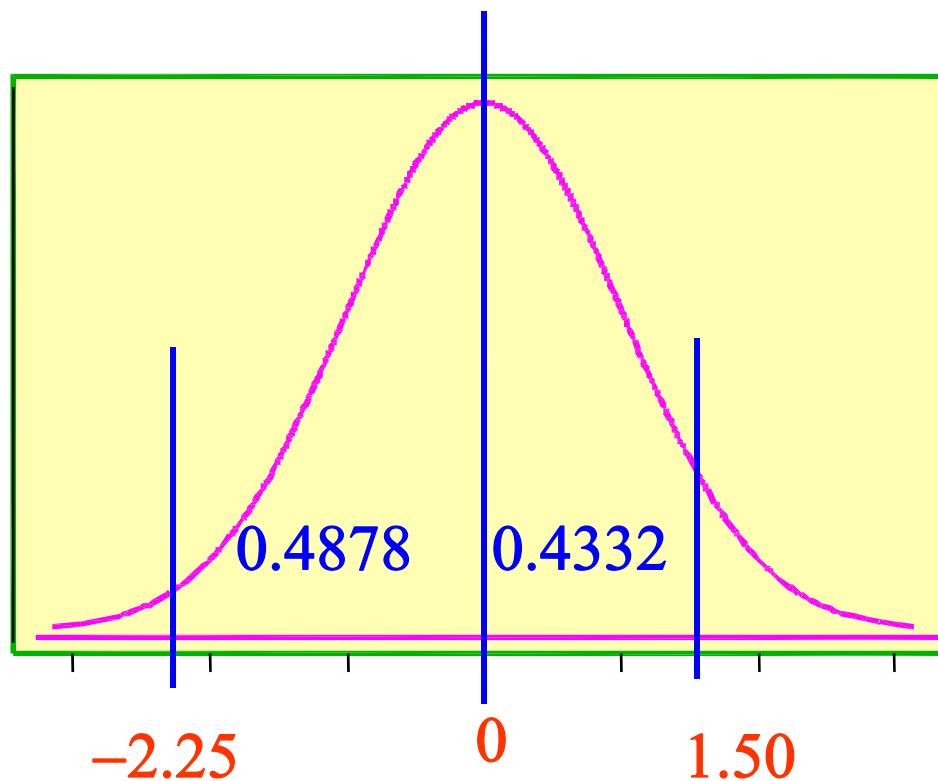
7-5 The Central Limit Theorem - Example

- The average age of a vehicle registered in the United States is 8 years, or 96 months. Assume the standard deviation is 16 months. If a random sample of 36 cars is selected, find the probability that the mean of their age is between 90 and 100 months.

7-5 The Central Limit Theorem - Example

- The standard deviation of the sample means is $\sigma/\sqrt{n} = 16/\sqrt{36} = 2.6667$.
- The two z-values are
 $z_1 = (90 - 96)/2.6667 = -2.25$ and
 $z_2 = (100 - 96)/2.6667 = 1.50$.
- Thus
 $P(-2.25 \leq z \leq 1.50) = 0.4878 + 0.4332$
 $= 0.921$ or 92.1%.

7-5 The Central Limit Theorem - Example



7-6 The Normal Approximation to the Binomial Distribution

- **The normal distribution is often used to solve problems that involve the binomial distribution since when n is large (say, 100), the calculations are too difficult to do by hand using the binomial distribution.**

7-6 The Normal Approximation to the Binomial Distribution

- The normal approximation to the binomial is appropriate when $np \geq 5$ and $nq \geq 5$.
- In addition, a correction for continuity may be used in the normal approximation.

7-6 The Normal Approximation to the Binomial Distribution

- A **correction for continuity** is a correction employed when a continuous distribution is used to approximate a discrete distribution.
- The continuity correction means that for any specific value of X , say 8, the boundaries of X in the binomial distribution (in this case 7.5 and 8.5) must be used.

7-6 The Normal Approximation to the Binomial Distribution - Example

- Prevention magazine reported that 6% of American drivers read the newspaper while driving. If 300 drivers are selected at random, find the probability that exactly 25 say they read the newspaper while driving.

7-6 The Normal Approximation to the Binomial Distribution - Example

- Here $p = 0.06$, $q = 0.94$, and $n = 300$.
- Check for normal approximation:
 $np = (300)(0.06) = 18$ and
 $nq = (300)(0.94) = 282$. Since both values are at least 5, the normal approximation can be used.

7-6 The Normal Approximation to the Binomial Distribution - Example

- (continued) $\mu = np = (300)(0.06) = 18$ and
 $\sigma = \sqrt{npq} = \sqrt{(300)(0.06)(0.94)} = 4.11.$
- So $P(X = 25) = P(24.5 \leq X \leq 25.5).$
- $z_1 = [24.5 - 18]/4.11 = 1.58$ and
 $z_2 = [25.5 - 18]/4.11 = 1.82.$

7-6 The Normal Approximation to the Binomial Distribution - Example

- (continued) $P(24.5 \leq X \leq 25.5)$
 $= P(1.58 \leq z \leq 1.82)$
 $= 0.4656 - 0.4429 = 0.0227.$
- Hence, the probability that exactly 25 people read the newspaper while driving is 2.27%.